A New Form of Understanding Maps: Multiple Representations with Pirie and Kieren Model of Understanding

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Abstract

The purpose of this study was to improve the mapping feature of the Pirie and Kieren model in the light of the use of multiple representations, to increase depicting power of the maps produced. As one of the qualitative research methods, case study design was used. This study was conducted with a sixth grade student at a public primary school in Turkey. Activity sheets, self-evaluation forms, and journals were used to collect data in addition to the semi-structured interview. The data collected from the interview were used to depict understanding maps of the student. The data from activity sheets, student journals, observations, and self-evaluation forms were used to strengthen the findings from the interview. This study showed that there was a relationship between the students' preference on the use of different type of representations and attained understanding level of multiplication of fractions. Moreover, it was seen that there was a relationship between question type and the students' use of representations.

Introduction

The words "I just don't like math", "I'm just no good at mathematics," "I hate mathematics in school", and "I don't understand mathematics" are frequently heard throughout school corridors. Students often do not understand the concepts in mathematics, and are far from understanding the actual meaning of those mathematical concepts. Because mathematical concepts and procedures are all constructed in the minds (Steffe, 2001, 2004; Thompson, 2003; von Glasersfeld, 1983), understanding mathematics can be difficult. This is an important issue, and it is always a big concern to assess students' understanding of mathematical concepts (Pirie & Kieren, 1989; Skemp, 1976). The assessment of students' understanding of mathematical concepts is not an easy task and thus, only limited parts of students' understanding can be assessed (Sierpinska, 1994).

Bruner (1960) came up with a distinct and clear definition of understanding, which described understanding as a product of thinking. However, his description of understanding was too broad and was for education in general, not for mathematical understanding specifically. Through the years, researchers have proposed several theories to describe learners' understanding of mathematical concepts. Skemp (1976) was one of the researchers who theorised mathematical understanding, and categorised understanding similar to Bruner (1960). According to Skemp, understanding includes two categories, known as relational and instrumental understanding. However, researchers argued about Skemp's approach to understanding. These arguments were mainly focused on how well that theory really explains understanding (e.g. Byers & Herscovics, 1977; Tall, 1978). This resulted in a revised version of Skemp's understanding theory. The revised theory contains two additional understanding categories: logical (Skemp, 1979) and symbolic (Skemp, 1982) understanding. Later, several other researchers followed the ideas of Skemp and proposed a number of understanding theories with different categories (Bergeron & Herscovics, 1988; Byers & Herscovics, 1977; Schroeder, 1987).

Pirie (1988) had doubts about categorizing students' understanding of mathematical concepts. Instead, she put emphasis on the processes. In her study, she discussed the idea of having several levels in the process of growth of understanding. A year later, Pirie and her colleague Kieren published the description of their theory (Pirie & Kieren, 1989). Pirie and Kieren (1994) described this theory of the growth of mathematical understanding as "a whole, dynamic, leveled but non-linear transcendently recursive process" (p. 166). As stated in Pirie and Kieren (1994), this is based on the constructivist definition of understanding detailed by von Glasersfeld (1983).

The Pirie-Kieren Theory of Understanding provides a framework to analyse people's growth of understanding by examining the move forward or backward from one level to another (Warner, 2008). These levels are embedded and illustrated with eight rings. Each of these rings represents different levels of understanding, which can be achieved by any person on any topic (Pirie & Kieren, 1994). From the inner most level to the outer most level, we can list the levels of understanding as follows: primitive knowing, image making, image having, property noticing, formalising, observing, structuring, and inventising. Primitive knowing is the first level of the process of understanding. This level does not imply low mathematical understanding, and is defined as a starting point of understanding by Pirie and Kieren (1994). This level indicates learners' prior understandings. The second level is the image making level where the learners are expected to make distinctions between prior knowing and use of it in new situations (Pirie & Kieren, 1994). At this level of understanding, learners do something mentally or physically to gain an idea about a concept. If a learner constructs an image about a topic in the mind, we can say that the learner is at the image having level (Pirie & Kieren, 1994). At this level, single-activity images are replaced with mental images. This means that learners do not need to perform physical activities when they deal with mathematics (Pirie & Kieren, 1992, 1994). At the fourth level, the properties of the constructed image are identified. This level is called property noticing. At this level, the learner can manipulate or combine properties of one's image to have context specific relevant properties (Pirie & Kieren, 1994). At the formalising level (fifth level), a method, rule, or property is generalised from the properties. Learners are expected to draw a method or common quality from the properties of previously hold images (Pirie & Kieren, 1994). At this level, class-like objects are developed from noticed properties. The learner verbalises and expresses about the formalised concept in observing level. In this level, learners can reflect on formal activities and express these as theorems (Pirie & Kieren, 1994). At the level of structuring, the learner organises his/her formal observations and deals with them as a theory (Pirie & Kieren, 1994). This is the level for formulating theories. The outermost level of the understanding is inventising, where the learner is expected to invent a new concept. The learner has a full understanding and he/she may create new questions, which lead him/her to a new concept (Pirie & Kieren, 1994)

The Pirie-Kieren Theory of Understanding acts as a powerful lens to observe the growth of understanding in a learning activity (Manu, 2005; Meel, 2003; Pirie & Kieren, 1989). It allows observing the growth of understanding of a specific person or a specific group (Pirie & Kieren, 1994). This theory does not take a photo of an instance, rather it allows depicting growth of understanding over a time period including minutes, hours, days, weeks, or even years (Manu, 2005; Pirie & Kieren, 1989). Pirie and Kieren (1989) created a technique to depict a persons' growth of understanding. They called this technique "mapping". Pirie and Kieren offered mapping as a presentation method for understanding. Pirie and Kieren (1994) stated their purpose of having this method as:

> *using the layered pictorial representation of the model, we aim to produce in diagrammatic form a 'map' of the growth of students' understanding as it is observed. This last phrase, 'as it is observed', is important because we make no claims as to what might have gone on 'in the students' heads'* (p. 182).

However, the mapping feature of the theory was not used widely among researchers, which may be because the current status of mapping is not considered informative or usable. Therefore, those who decided to use this feature changed it slightly (Borgen, 2006; Borgen & Manu, 2002; Manu, 2005; Meagher, 2005; Pirie & Martin, 2000; Towers, 1998).

The most radical modification was made by Towers (1998). She changed the overall appearance of the map and used parallel layers instead of having embedded circles (see Figure 1). This allowed her to depict the most complex and longer understanding processes; however, the map still functioned in a similar way to the one developed by Pirie and Kieren (1989).

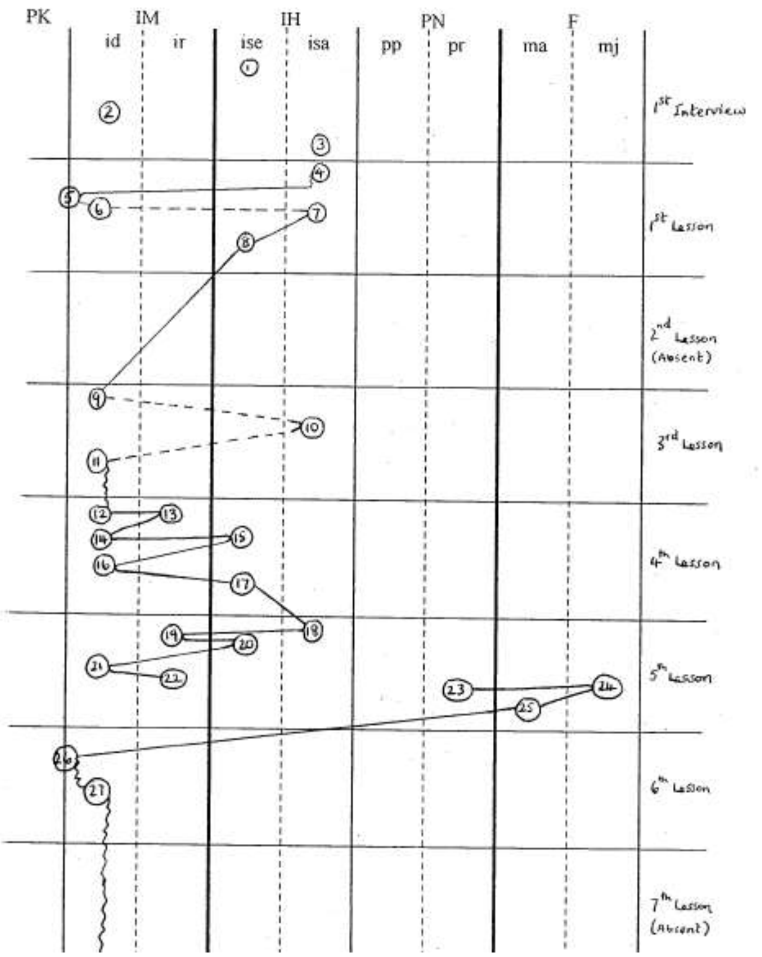

Figure 1: Parallel layered map from Towers (1998, p. 127)

Meagher (2005) used the same original layout, but he simplified the drawings with simple arrows and he counted subsequently each activity point at the same level. This map can be seen in Figure 2.

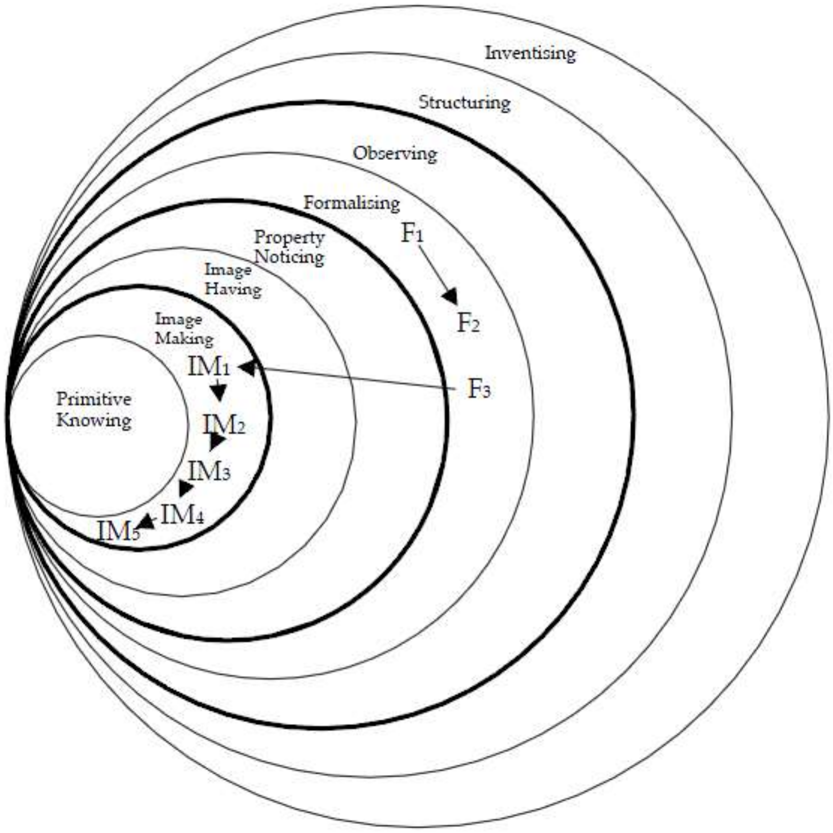

Figure 2: Simplified map from Meagher (2005, p. 151)

In the current study, the researcher paid attention to students' use of representations to examine their understanding in multiplication of fractions. Cai (2004) claims that studies about representation usages bring important ideas about what representations should be used in a mathematics course. According to von Glasersfeld (1983), use of representations differs for each learner. Similar results were found in similar studies (Cai, 2000a; Cai & Hwang, 2002). It was found that the students who used symbolic representations had greater achievement scores than those who used verbal and pictorial representations (Cai, 2000b). Moreover, this is also the case with textbooks and the representations used in them (Son & Senk, 2010). However, it should be noted that none of the representations are superior to the one other. A particular representation can be more suitable than the other for a specific mathematical concept (Ball, 1993).

There are also several studies that investigated the multiple representations with the light of Pirie-Kieren theory. Study of Droujkova (2004) showed that using symbols or abstract descriptions could support learning indirectly. Moreover, folding back to image making level to support learning requires the use of more tangible representations. Property noticing and formalising levels corresponded to more abstract representations. Similarly, Wilson and Stein (2007) found that the students' use of more abstract representations becomes more frequent toward the outer level of understanding.

The primary focus of the current study is to find an effective and accurate way in which we can depict the understanding process in mathematics. With that focus, the researcher discussed the idea of tracking and depicting the use of representations of the students through the process of understanding. As can be seen in the literature, the use of representations makes mathematical communication possible (Kilpatrick, Swafford, & Findell, 2001). Therefore, in order to accurately analyse the learners' understandings, the learners' use of multiple representations should be carefully examined (Cramer, Wyberg, & Leavitt, 2008;   , 2003; Taber, 2001). This also gives an opportunity to see which representations should be used to convey necessary information to the students so that they will have an understanding of mathematical concepts.

Hiebert and Carpenter (1992) stated that understanding results from the interaction between mental and external representations. According to them, better connections between these two types of representations meant better understanding. However, students' mental representations could not easily be observed without analysing their use of external representations (Goldin, 1998). Therefore, in order to analyse understanding we need to concentrate on the student's use of external representations.

In the current study, the researcher explored a sixth-grade student's understanding of multiplication of fractions by revealing actions to indicate the understanding. Uses of representations were analysed to have clear picture of his understanding process. The researcher examined how we can further improve the Pirie-Kieren Theory of Understanding with the use of representations, and tested the effectiveness of this modified version. Thus, the purpose of this study was to improve the mapping feature of the current theory, to increase the depicting power of the maps produced. This study extends the current body of literature using the Pirie–Kieren Theory of Understanding by applying the theory to analyse students' uses of multiple representations.

Research questions

The research questions, which underpin this study, are:

- What representations do students use to express their multiplication of fraction ideas at each understanding level in terms of Pirie and Kieren's model?
- How can we improve the mapping feature of Pirie and Kieren's theory with the use of multiple representations?

Method

Design of the study

The case study design, one of the methods in qualitative research, was used in this study (Merriam, 1998; Patton, 2002, p. 447). The case was the process of a student's efforts towards understanding concepts related with multiplication of fractions.

Setting and participants

The study was conducted at a small sized public primary school in Turkey. The school had two floors and a garden. There were 10 classrooms, a kindergarten, a computer laboratory, and science laboratory in the school. Each classroom had a projector, and a PC. In the school, concrete materials were available for mathematics courses. The teachers followed the national elementary mathematics curriculum; and, sixth grade students attended mathematics courses four hours per week. There were two mathematics teachers in the school.

Purposive sampling was used to choose the participant in the study. A male student was selected for this study. The selection of the student was based on two criteria: (1) The researcher's impression of the student's willingness to respond to questions; and (2) The student's mathematical ability. The first criterion was to maximise responsiveness during the interview. The second criterion was decided according to the previous grades of the student. The participant was 12 years old; the name Ali was given as a pseudonym. He was the best student in his class, and he was very active in his lessons, always asking questions.

Data collection

The data were collected using interview, students' journals, activity sheets, observations, self-evaluation forms, and field notes. Table 1 gives the time line of the data collection process. In addition to the information given in the table, the class was observed throughout multiplication topics.

Table 1: Data collection time line

Week	Measure
1	Consent forms received
1-3	Instruction done/Activity sheets given/ Students' self-evaluation forms received
1-4	Students' journals received
5	Interview conducted

The fraction interview instrument, activity sheet, student journal, and student self-evaluation form were developed by the researcher. The interview was carried out after school hours. Paper and pens were provided to the student, as were concrete materials related to fractions such as fraction strips, transparent fraction cards, small cubes, beans and chickpeas. The researcher videotaped the interview and the student's notes were kept. Later, these video records were transcribed. The student was informed that his answers were not judged, as being correct or incorrect, and that answers would not affect his course grades. He was also informed that there was no defined duration in order to answer each question. Follow-up questions were asked according to the answers of the student. Some of the follow-up questions were as follows: "Can you explain this in detail?", "Why did you think like that?", "Can you give one more example to this situation?", and "How did you come up with this idea?" The purpose of these follow-up questions was to probe the student's answers to the questions and the reasons for the particular answer.

In order to analyse the students' level of understanding, there were several actions for each question, and specified for each level of understanding, that corresponded to the Pirie and Kieren's theory of understanding. These actions can be seen in Table 2 with respect to level of understandings.

Table 2: Levels of understanding and corresponding actions

Level of understanding	**Action associated with fractions**
Primitive Knowing	The student is assumed to know: • how to do multiplication operation with natural numbers, • what is the meaning of multiplication operation with natural numbers, • that multiplication is a form of repeated addition in natural numbers.
Image Making	The student is able to: • model multiplication of two fractions by using transparent fractions cards, chick beans, beans, fraction sticks etc., • draw figures of multiplication of two fractions, whole number by a fraction, • do paper folding activities for multiplication of fractions.
Image Having	The student is able to: • explain his/her modeling/drawing of multiplication of two fractions, whole number by a fraction.
Property Noticing	The student is capable of: • multiplying a whole number by a fraction, • multiplying a fraction by a fraction (proper and improper fractions), • multiplying mixed numbers, • noticing commutative and associative property of multiplication of fractions.
Formalizing	The student is capable of recognizing that: • multiplying a whole number by a fraction means taking a fractional part of this whole number • multiplying a whole number by a fraction can be explained with multiplication as a form of repeated addition. • multiplying a fraction by a fraction means taking a fractional part of this fraction which means taking a part of a part.
Observing	The student is able to: • connect that multiplication of fractions are used in different mathematic topics such as proportion, percentages, time measures etc. and give examples for them.
Structuring*	The student is capable of : • recognising the relationship between fractions and rational numbers. • developing theories with multiplication of fractions.
Inventising*	The student is capable of : • creating new concepts as a result of fully understanding multiplication of fractions.

* The participants of the current study were not expected to reach these levels

These actions were derived and modified from some of Pirie and Kieren's studies about fractions (Kieren, 1988, 1999; Kieren & Nelson, 1978; Kieren & Pirie, 1991; Pirie & Kieren, 1992), as well as some of the studies that applied the Pirie-Kieren Theory of Understanding as theoretical lens to their studies (Meel, 1995; Tsay, 2005). There was no question that could help us to trace student's understanding toward the structuring and inventising levels, as these levels required more sophisticated questions that were not suitable for the objectives of multiplication of fractions in the sixth grade mathematics curriculum. Even the students from Pirie and Kieren's study did not achieve higher levels such as structuring and inventising (Pirie & Kieren, 1994). Students were allowed to use any type of representation-verbal, concrete, visual and symbolic- through each question.

The activity sheet was used as a supplementary instrument for the triangulation of the data in the current study. There were 14 questions in the activity sheet related with modeling of multiplication of fractions, real life word problems, and symbolic questions about multiplication of fractions. This activity sheet was completed by the students during the lessons relating to the multiplication of fractions.

Student journals were also used as a supplementary document for the triangulation of the data in the current study. Students' Self-Evaluation Forms were another supplementary instrument for the triangulation of the data in the current study. The purpose of these forms was to allow students to evaluate themselves and see their deficiency with the multiplication of fractions. Moreover, the lessons and interview were recorded in order to more deeply understand the entire procedure. The videotapes were used as an observation tool in this study. The purpose of observations during the lessons related with multiplication of fractions was to validate the data collected from the interview. Field notes refer to written notes derived from data collected during observations and the interview. The field notes collected during the study were used for the triangulation of the data collected from interview.

Data analysis

Data coding analysis was used in this study. The coding was done according to the levels of understanding and corresponding actions as shown in Table 2, and also according to the student's use of multiple representations. The representation type was coded according to the student's use during each question. Moreover, the representation type and the level of understanding presented by a student for each question were numbered in order to draw an understanding map for that student. So, there could be more than one number in a question depending on the student's answer. The abbreviations were used to show the levels and representation types. There were also supplementary data sources such as activity sheets used during the lessons, student journals, and self-evaluation forms, field notes and classroom observations were used for validating the data collected through fraction interview instrument. Some of the answers of the student on the activity sheet, student journal, and self-evaluation forms were used to support the coded data in the interview.

Results

The data from the interview, observations, and other sources (activity sheets, journals, self-evaluations) were used to analyse understanding and uses of representations at each level of understanding, to find emergent themes concerning types of representations and understanding levels in which representations were used. Understanding maps were generated with these. The map demonstrates the associations between the types of representations used by the participants and the levels of understanding.

The representation types, understanding levels, and movement from each of the levels were coded, as seen in Table 3, to analyse the data.

Table 3: Coding scheme used in analyses

Code	Description
P	Primitive knowing
IM	Image making
IH	Image having
PN	Property noticing
F	Formalizing
O	Observing
S	Structuring
I	Inventising
IMtoP	Folding back from image making to primitive knowing
IHtoP	Folding back from image having to primitive knowing
IHtoIM	Folding back from image having to image making
PNtoIM	Folding back from property noticing to image making
PNtoIH	Folding back from property noticing to image having
PNtoP	Folding back from property noticing to primitive knowing
FtoIH	Folding back from formalizing to image having
FtoIM	Folding back from formalizing to image making
FtoP	Folding back from property noticing to primitive knowing
RV	Using visuals (drawing, diagrams) as representations
RW	Using spoken or written words as representations
RC	Using concrete materials as representations
RS	Using symbols as representations

Preliminary analyses of the supplementary data showed that Ali used symbolic representations in almost every stage of the lesson. He tried to explain his answers with symbols as well. Therefore, we can say that, in the instructional sequence Ali used symbols, drawings, and verbal language, less than he used symbolic representations on the activity sheets given. This trend of using symbols continued throughout the observation (see Figure 3). This initial map outlined Ali's use of the representation types related with his understanding levels. Numbers of arrow from understanding levels to representation types were given representatively. The numbers at the right corner of the map showed the frequency of representation types he used at each understanding level.

Ali's initial map showed that he used four concrete material and two visual representations at image making level; four symbolic and verbal representations at image having level; two symbolic and one visual representation at property noticing level; two symbolic representations at formalizing level; and a symbolic representation at observing level. It seemed that Ali mostly used symbolic and verbal representations at outer levels; concrete materials and visual representations at inner levels.

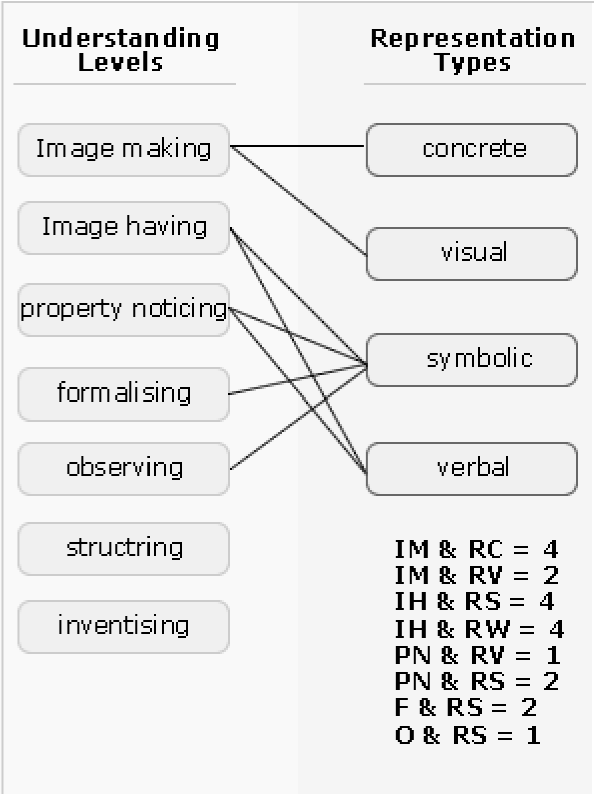

Figure 3: Activity levels vs. type of representations from Ali's first observation

As stated before, primary data sources for analysing understanding was the data gathered from the semi-structured interview. The first question was related to trying to find out what comes to one's mind about fractions. The purpose of this question was to find out the relationship between multiplication and repeated addition, while obtaining the fraction. Ali was asked to find out how he could obtain the fraction $\frac{4}{5}$. He showed his awareness for figuring out a fraction by drawing a figure and using a concrete material, as shown in the following excerpts. At this stage, Ali demonstrated his ability to operate at image making level (IM) when dealing with the concept of fractions. He used visual representations (IM) (RV) [1] and concrete representations (IM) (RC) [2] to represent the fraction $\frac{4}{5}$.

> ALI: *The fraction* $\frac{4}{5}$ *means dividing a whole into 5 equal parts and taking 4 parts of it (He was drawing a rectangle at the same time, see Figure 4) (IM) (RV) [1] I can also show it with fraction tiles (IM) (RC) [2] ……I can also show it with chick peas (RC). If one chick pea is taken as* $\frac{1}{5}$, *then I take 4 chick peas in order to show the fraction* $\frac{4}{5}$.

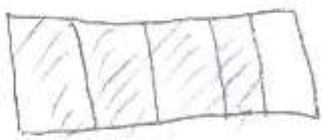

Figure 4: Ali's visual representation of $\frac{4}{5}$

While Ali was further explaining the fraction, he demonstrated the fraction $\frac{4}{5}$ with mathematical expressions. At that stage, he explained the fraction as operations. He started with addition of the fractions and concluded that multiplication is a form of repeated addition, which showed he was operating at the formalizing level (F). He used symbolic representations (RS) [3]. (F) (RS) [3]. More evidence for this was also seen on his activity sheet. He tried to explain the relationship between multiplication and repeated addition.

In the second question, ingredients to make a cake for five people were given and the proportion of each ingredient was requested for making a cake for fifteen people. It was related with finding the multiplication of a whole number by a proper fraction, an improper fraction and by a mixed number. It also emphasized the relationship of multiplication and repeated addition. Ali showed an understanding of the multiplication of a fraction by a whole number, as well as multiplication of a mixed number by a whole number. He multiplied the numerators of the fractions and multiplied the denominators of the fractions, and placed the product of the numerators over the product of the denominators. All aforementioned evidences proved that he positioned at property noticing level (PN), since he examined his knowledge about multiplication of fractions for relevant properties. He used symbolic representations (RS) in order to demonstrate these properties as seen in the following excerpts. [4]

> ALI: *It is for five people, it'll be multiplied by three for the cake prepared for 15 people. I will find the amount of vegetable oil by multiplying $\frac{1}{2}$ with 3. An integer can be considered to be a fraction with a denominator of 1. There is no cancelling for this operation…I multiply the numerators, it is 3 and multiply the denominators, it is 2. Then the product is $\frac{3}{2}$ which equals to $1\frac{1}{2}$ cup of vegetable oil.* (see Figure 5) (PN) (RS) [4]

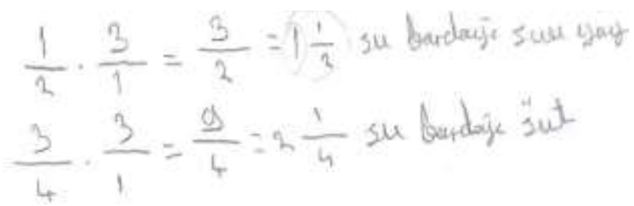

Figure 5: Symbolic representation used by Ali to show multiplication of a fraction by a whole number

Ali did the other multiplication ($\frac{3}{4} \times 3$) the same as in the previous excerpts, in order to find the amount of milk used. He also multiplied a mixed number by a whole number to find amount of the sugar and flour being used in the cake. He converted each mixed number to an improper fraction and then did multiplication. He positioned at property noticing level (PN) and used symbolic representations (RS) [4]

Ali finally found the proportion of eggs and baking powder by just multiplying them by 3. He had primitive knowledge (PK) about fractions [5] and he used symbolic representations (RS). For instance, he was asked what he understood from $2\frac{1}{3}$ cup flour, he responded "2 cup flour and a cup with an amount nearly filled $\frac{1}{3}$ of it". He drew the figure to illustrate his explanation and used visual representation (See Figure 6) (RV)

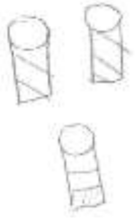

Figure 6: Visual representation used by Ali to show a fraction

Ali talked about the half cup or $\frac{1}{3}$ cup. He knew the meaning of denominator and the numerator of a fraction. He also identified an integer can be considered to be a fraction with a denominator of 1 and he defined improper fractions and mixed numbers and he was able to convert them to each other. Ali also talked about the connection between addition and multiplication when a fraction was multiplied by a whole number. He demonstrated his ability to verify multiplication was a form of repeated addition, which showed he positioned at formalizing level (F). He used symbolic representations (RS) for this verification [6]. More evidence can be found in Ali's journal (see Figure 7) and activity sheet (see Figure 8).

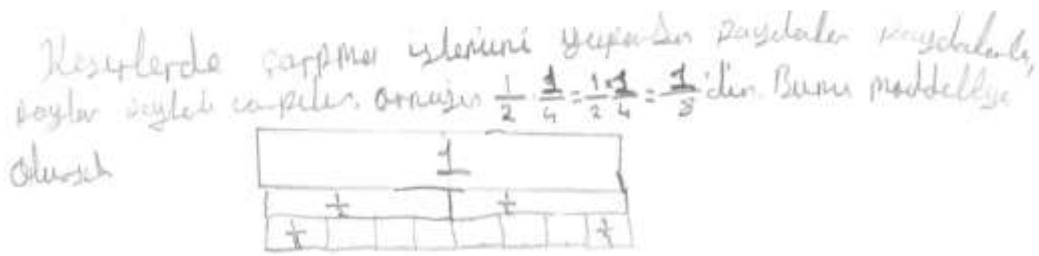

Figure 7: Symbolic and visual representations used by Ali to show multiplication of a fraction by fraction in the journal

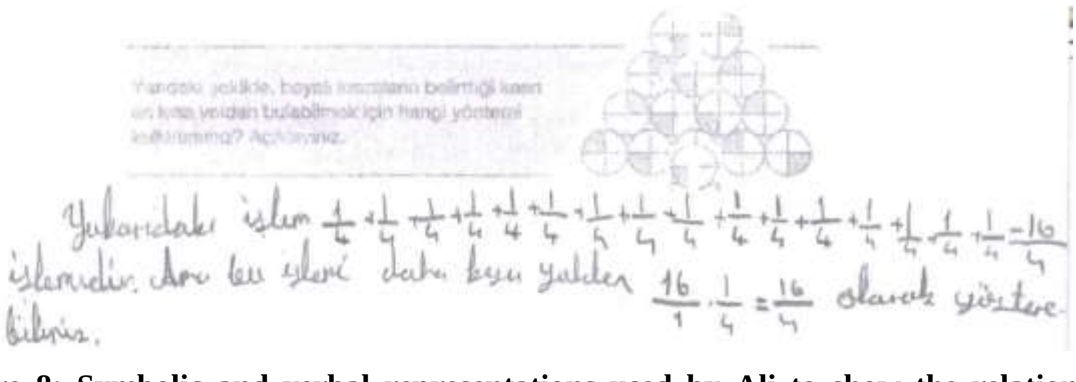

Figure 8: Symbolic and verbal representations used by Ali to show the relationship between multiplication and repeated operation in the activity sheet

The third question was a word problem relating to finding a part of a whole number. Ali was asked to find $\frac{4}{5}$ and $\frac{2}{3}$ of 45. Ali verified his knowledge of multiplying a fraction by a whole number in different settings, which pointed at his ability to operate at property noticing level (PN). He used symbolic representations (RS). See the following excerpts.

ALI: $\frac{4}{5}$ *of teacher's book means multiplying* $\frac{4}{5}$ *by 45. I give 1 to the denominator of 45 in order to do this multiplication since a whole number can be considered to be a fraction with a denominator of 1. I can simplify 45 and 5 by dividing them with 5. Then, I find the product by multiplying the numerators together and denominators together and find the product 36. So, the teacher has 36 novels*. (see Figure 4.11.) (PN) (RS). [7]

RESEARCHER: *Why did you do multiplication here?*

ALI: *I found* $\frac{4}{5}$ *of 45. I can divide 45 by 5 and multiply the result by 4. It is the same as multiplying the* $\frac{4}{5}$ *by 45. It means adding 45 times* $\frac{4}{5}$ *together.* (PN) (RW) [8].

Ali knew the meaning of multiplication and he was aware that multiplication was a form of repeated addition. He also knew that multiplying a whole number by a fraction meant finding a part of a whole. More evidence can also be seen in his activity sheet and his journal. These evidences helped to determine that he was at formalizing level (F). He used verbal representations (RW) for these explanations [8]. He did the same operations for finding $\frac{2}{3}$ of 45 and found 30 for the doctor's novels (see Figure 9). He also modeled $\frac{4}{5}$ of 45 books (see Figure 10). He verified his knowledge of multiplying a fraction by a whole number. See the following excerpts.

ALI: *I draw a square and divide it into 5 equal pieces and take 4 parts of it. I divide 45 by 5, I find each part has 9 books inside. I take 4 parts which means 4 times 9 equal to 36 books* (see Figure 10). (RW)

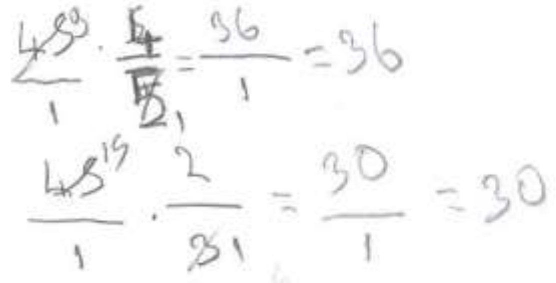

Figure 9: Symbolic representations used by Ali to show multiplication of a whole number by a fraction

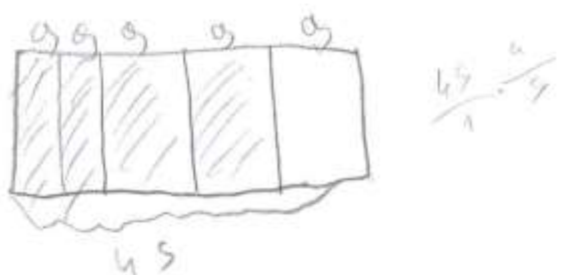

Figure 10: Visual representation used by Ali to show a part of a whole

The fourth question was a word problem related to the multiplication of a mixed number by a whole number. It was related to finding a part (a mixed number) of a whole number. The number of egg cartoons in a grocer and the number of eggs inside each egg cartons were given. Ali was asked how many eggs there were in all of the cartons, combined. He demonstrated his knowledge of connecting fractions with natural numbers. He knew what a half fraction meant, and used this knowledge by connecting the multiplication of fractions with natural numbers. He used symbolic representations (RS) while solving this question. He positioned at formalizing level (F) since he was aware of the connection between natural numbers and fractions [9]. He had an idea about the fractions of quantities.

Ali was asked if he could solve this question in different way, and he used fraction multiplication. He converted the mixed number to an improper fraction and then did multiplication, finding the same result. He used symbolic representations (RS) as in the following excerpts. Moreover, he did similar operations on his activity sheet during the lesson and his journal. He positioned at property noticing level (PN) since he just did multiplication [10]. He also expressed what an improper fraction and a mixed number.

> ALI: $\frac{27}{2}$ *is multiplied by* $\frac{12}{1}$*. I simplify 12 and 2. I find the result 162, the same as the first result… This multiplication means adding 12 times* $\frac{27}{2}$ *. There are 12 eggs in each package and we should use multiplication in order to find the number of eggs in 13 and a half package* (see Figure 11). (RS)

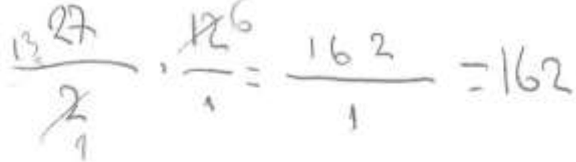

Figure 11: Symbolic representations used by Ali to find a mixed number of a whole

The fifth question was related to the modeling of multiplication operation with a whole number and a fraction. It was related to the modeling times of fractions. A shape was given as a whole in rectangular units, and Ali was asked to model a multiplication operation in rectangular units by using this whole. Ali tried to model the multiplication operation. He first tried to model it by using cubic blocks and then tried to draw and express it on the given shape (see Figure 12). It initially seemed he modeled correctly. He positioned at image having level (IH). More evidence can also be seen in his journal, and activity sheet for modeling and knowing the meaning of the multiplication of a whole number by a fraction (see Figure 14). He used concrete materials as representation (RC) while modeling [11]. But, while drawing it in rectangular units (see Figure 13), it seemed that he did not notice that he modeled it correctly as in the following excerpts (RV).

> ALI: *This equals to* $\frac{1}{6}$ *(he divided 3 rectangular units into 6 equal pieces and hatched a piece in order to show* $\frac{1}{6}$*). If 4 of them are put side to side, the result of this operation (he is counting) 1,2,3,4,…..24. Wait a minute 24. We should not find 24 since the result of this multiplication is* $\frac{4}{6}$*. 4 times* $\frac{1}{6}$ *are added together. I found the result* $\frac{4}{24}$ *from the model which means 24 times* $\frac{1}{6}$ *come together. But, this does not give the same result* (see Figure 12) (RV).

Although he initially modeled the operation correctly, he could not notice this. He made several attempts, but he was unsuccessful since he thought 4 wholes instead of one whole and divided 4 wholes into 6 equal pieces. He knew multiplication is a short way of repeated addition, but he could not connect this knowledge with modeling. So, he had difficulty in finding the correct modeling. All the aforementioned evidence pointed at his ability to operate at image making level (IM). He folded back to image making level (IHtoIM). He used symbolic representations (RS) for expressing his model [12].

> ALI: *I divided the whole shape into 6 equal pieces… The denominator should not be added… the denominator must be the same. So, it equals to $\frac{4}{6}$ which means 4 times $\frac{1}{6}$ added together. It is the short way of repeated addition* (see Figure 13). (RS and RV).

Figure 12: Modeling of the operation 4 times of $\frac{1}{6}$ in rectangular units

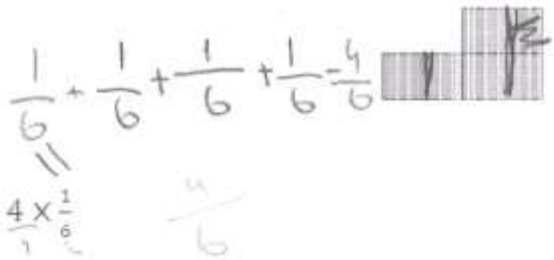

Figure 13: Symbolic and visual representations used by Ali to find 4 times of $\frac{1}{6}$

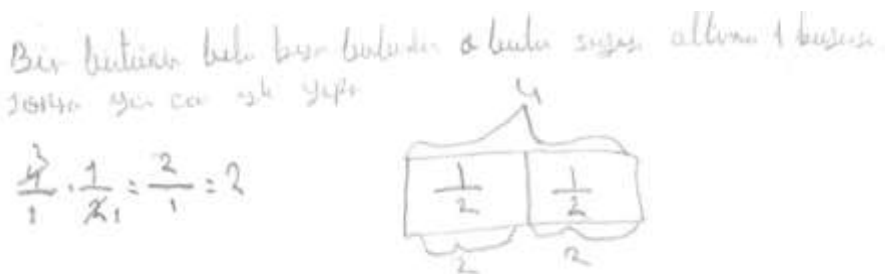

Figure 14: Symbolic and visual and verbal representations used by Ali to show multiplication of a whole number by a fraction (in Ali's journal)

According to the results from the interview, the understanding map was generated. As you can see from Figure 15, seven activity points over 12 were at symbolic region and most of these activities were at property noticing and outer level. Moreover, this map confirms the initial analysis that shows the trends on using symbolic representations by Ali.

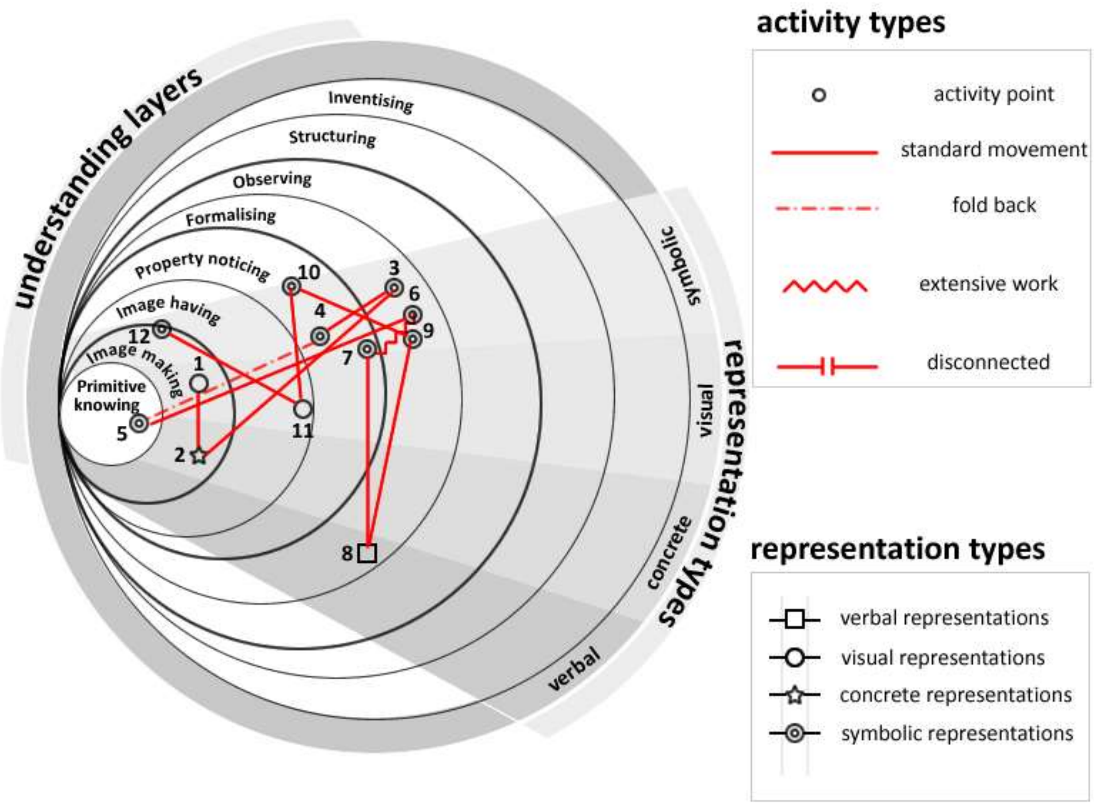

Figure 15: Understanding map of Ali

Discussion, conclusions, and implications

The Pirie-Kieren Theory of Understanding has an important claim to depict students' understanding processes over the time, with the help of the mapping feature proposed by Pirie and Kieren (1989). However, except Towers (1998), Borgen (2006), and Manu (2005), other researchers, who employed the Pirie-Kieren Theory of Understanding to analyse learners' understanding, did not depict it with the maps utilised by Pirie and Kieren (1989). Moreover, Towers (1998) and Manu (2005) slightly changed the mapping feature when they applied it to their study. They worked on the shape of the map, and used a table-like layout for the maps. This usage makes it easier to depict learner's understanding process with map; however, it is difficult to interpret it because it takes much more space, and it is hard to follow the stream. Moreover, as pointed out by Borgen (2006), this change is criticized in a way that it diminishes the embedded structure of the understanding proposed by Pirie and Kieren (1989). In the current study, the researcher applied new icons and design elements to make the maps more interpretable. Additionally, one more dimension, representation types, was added to the existing map. The new version of the map preserved its original embedded rings shape. The improved version of the map can be seen in Figure 15. Pirie and Kieren (1994) emphasize the power of mapping, and stated that drawing understanding maps allows us to clearly depict the growth of understanding.

Some of the studies showed that attending to the representations is crucial for communication and development of mathematical understanding (Kilpatrick et al., 2001; Taber, 2001). Furthermore, Pirie and Kieren also emphasize the importance of the use of representations in their studies (Droujkova, 2004; Wilson & Stein, 2007). Therefore, use of representations was added as a second dimension to the current mapping feature. Several other researchers also modified the mapping feature of the theory. The most significant change was made by Towers (1998). In the current study, Ali's understanding map was also depicted with the map suggested by Towers (1998) (see Figure 16).

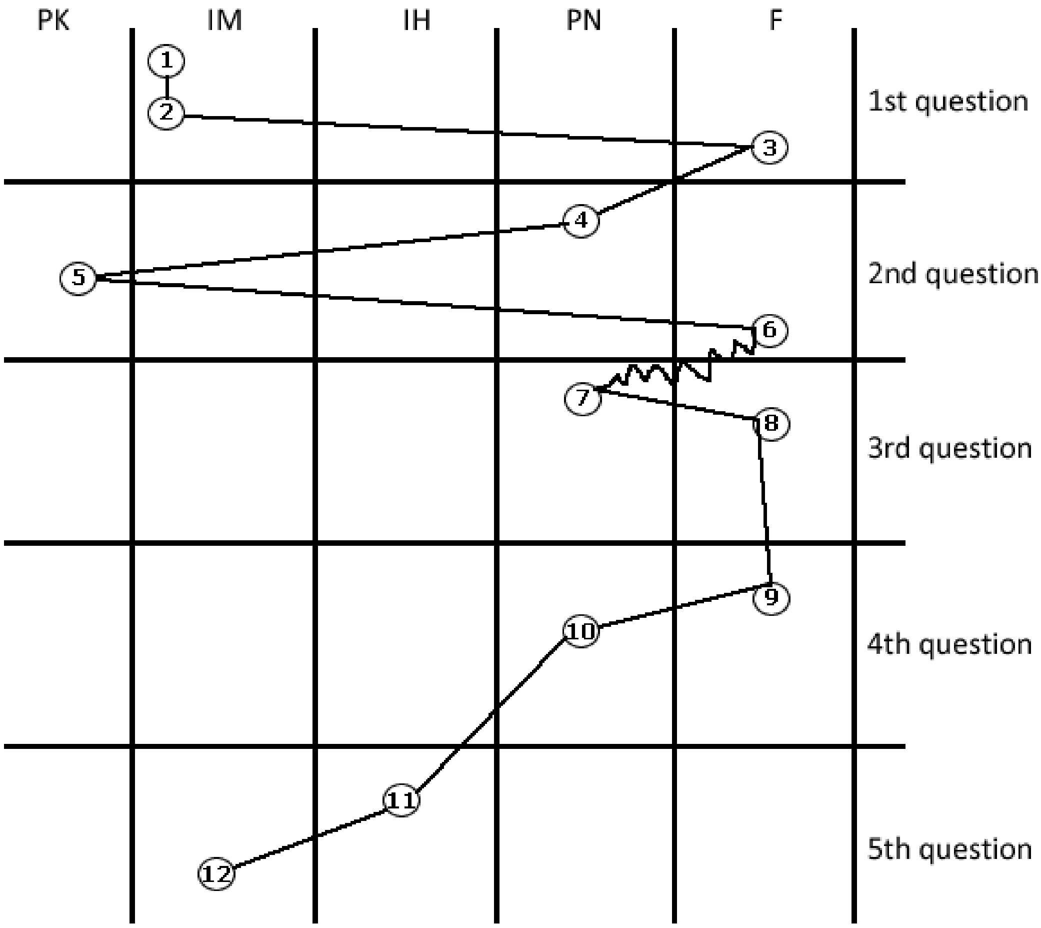

Figure 16: Tabulated growth of understanding of Ali

The mapping style proposed by Towers enables us to check details, question by question. This surely gives us a handy report to work with; however, when the number of questions increases, this table-like map also increases in length, making it hard to follow the stream through the observed sessions. However, Towers did not suggest a separate notation for folding back activities; therefore, the shape of the map does not give any information on whether that activity is folding back or not. Moreover, as pointed out by Borgen (2006), this layout does not represent the nature of the Pirie-Kieren Theory of Understanding that claims embedded eight levels of understanding.

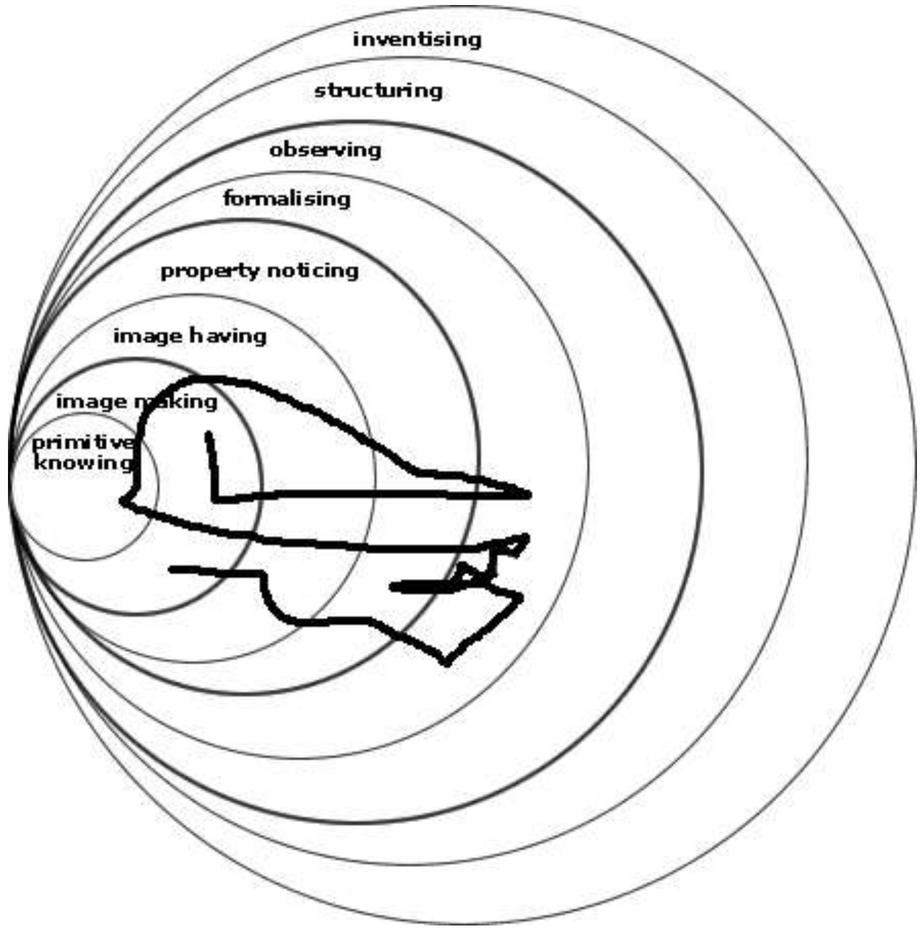

Figure 17: Original map of Ali's growth of understanding

You can see the growth of Ali's understanding depicted as with original layout in Figure 17. The map in Figure 16 does not offer more details than the maps in Figure 17. Nonetheless, it is much more easily interpretable than the mapping technique offered by Pirie and Kieren (1989)

We can say that the map produced in this study gives more insight about understanding process than the original and previously suggested other modes of maps. Incorporating the representation types to the maps, we transferred the power of communicating with representations to this depiction (Kilpatrick et al., 2001).

In addition, the results of the current study showed that students can solve the questions even without understanding. Therefore, it is not enough to follow just their responses to say something about their understanding in multiplication. This is also emphasised by Hiebert and Carpenter (1992, p. 89). However, tracking their use of representations, allowed the researcher to track the understanding as well. This issue also pointed out by other researchers (Taber, 2001). Tracking the students' use of representations with respect to their understanding level provides us valuable information on their understanding process.

The data support that there is a relationship between students' preference on the use of different type of representations and attained understanding level. The specific actions related with the use of representations can be seen in Table 4.

Table 4: Understanding levels using representations adapted for the Pirie-Kieren model for growth of mathematical understanding

Level of understanding	Activities & use of representations
Primitive knowing	Can use any representations to recall prior knowledge
Image making / image having	Can connect the representations to the problem being studied
Property noticing	Can connect the representations to mathematical meanings and can compare representation properties
Formalizing	Can describe mathematical meanings with the use of representations to generate patterns and algorithms. Mostly uses symbolic and verbal representations.
Observing	Can connect representations to a theorem. Mostly uses symbolic and verbal representations.

In the current study it was seen that there was a relationship between question type and the students' use of representations. This should be further investigated with experimental studies at different grade levels, using different topics. Moreover, this study concentrated at the topic of multiplication of fractions. The improvements of the understanding map should be tested using different topics. The current study offered a new technique to depict students' growth of understanding. This technique was tested with multiplication of fractions. Since different concepts can require different type of representations, this map should be tested on different topics as well.

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