TRANSITIONED CURVES FOR TRAMWAYS.

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In its original form this paper consisted of notes on one or two points in the practical application of transitioned curves to tramways, in which the author expressed the opinion that there was room for improvement, either in the direction of greater simplicity or uniformity. In order to make these notes more intelligible to those who have not previously had to do with these curves the paper has now been made a more general one. In doing this I have not touched on the Field work or the calculation of the normal types of curves, as these matters have been dealt with very thoroughly by Mr. T. Kennedy, Engineer in charge of Railway and Tramway Surveys, and by Mr. J. C. Try, B.E., of the Railway Survey Branch, whose papers are reprinted herewith. The thanks of the Society are due to these gentlemen and to the Institute of Surveyors for their permission to reprint the papers, also to Mr. C. J. Merfield of Melbourne Observatory, who has kindly consented to the publication of his transition table (Appendix C).

The subject of "The calculation of Tramway Curves" has been pretty thoroughly dealt with by several officers of the Railway Construction and Railway Survey Branches of the Department of Public Works. Mr. W. Shellshear and Mr. C. J. Merfield discussed the cubic parabola as a means of easing the circular curves on Railways; Mr. Merfield prepared Tables for both Railway and Tramway curves and showed how to locate the parabola by the use of these tables, and Mr. J. C. Try showed their practical application to all varieties of curves met with in the field.

For further information on the subject the following papers may be consulted :—

(1.) On a simple plan of easing Railway Curves; by W. Shellshear, P.R.S., N.S.W., vol. xxii, 1888, p. 89.

(2.) The Cubic Parabola as applied to the Easing of Circular Curves on Railway Lines; by C. J. Merfield, P.R.S., N.S.W., vol. xxix., 1895, p. 51.

(3.) Notes on the Cubic Parabola applied as a transition to small Tramway Curves; by C. J. Merfield, P.R.S., N.S.W., vol. xxxi, 1897, p. 56.

(4.) Tables to facilitate the location of the Cubic Parabola; by C. J. Merfield, P.R.S., N.S.W., vol. xxxiv., 1900, p. 281.

(5.) Tramway Curves; by J. C. Try, B.E., Surveyor, September, 1909.

Object of Transition to Curve.

The object of applying a "transition" or curve of varying radius to a circular curve is to allow the super-elevation of the outer rail (which is necessary to counteract the centrifugal force of the train or tram) to be applied gradually, so that at any point on the curve the elevation for a given speed will be suitable to the radius at that point. It is further desirable that the change of elevation should be uniform, in order to avoid the difficulty of bending rails in two planes at once.

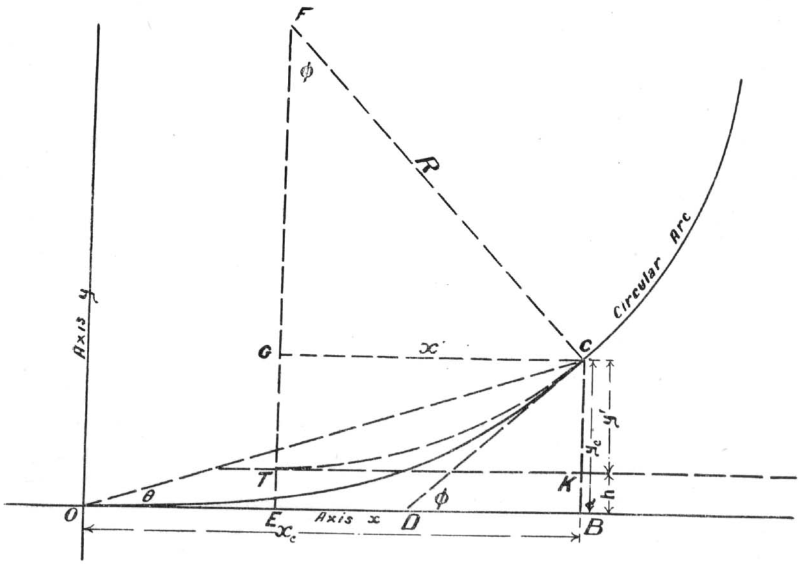

Fig. 1.

Determination of Suitable Curve.

To find the curve best suited to this purpose we have:—

$$\text{super-elevation (inches)} = \frac{\text{gauge (inches) x velocity}^2\text{ (m.p.h.).}}{1{\cdot}25 \text{ x radius (feet).}}$$

If $\frac{1}{e}$ be the rate of rise of the outer rail, the rise at any distance along the X axis will be $\frac{x}{e}$, so that

$$\frac{x}{e} = \frac{gv^2}{1{\cdot}25\rho} \text{ or } \rho = \frac{gv^2e}{1{\cdot}25x} = \frac{c}{x} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where c is a constant for any given velocity.

Now for any curve $\rho = \dfrac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \ldots \quad \ldots \quad \ldots \quad (2)$

or taking ds = dx $\dfrac{d^2y}{dx^2} = \dfrac{1}{\rho} = \dfrac{x}{c}$

Integrating twice :—$y = \dfrac{x^3}{6c}$ or writing $\dfrac{1}{m}$ for 6c

$$y = mx^3 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (3)$$

a cubic parabola.

It will now be shown how to find a parabola which may be applied to a curve of any given radius.

Differentiating we have

$$\frac{dy}{dx} = 3mx^2 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (4)$$

$$\frac{d^2y}{dx^2} = 6mx$$

so that $\rho = \dfrac{(1 + 9m^2x^4)^{\frac{3}{2}}}{6mx}$

writing tan ϕ for $\dfrac{dy}{dx}$ and taking R as the radius of the circular curve, we have at the point of contact (x_c, y_c)

$$R = \rho_c = \frac{(1 + \tan^2\phi)^{\frac{3}{2}}}{\tan\phi} \times \frac{x_c}{2} = \frac{x_c}{2}\,\frac{\sec^3\phi}{\tan\phi} \quad \dots \quad \dots \qquad (5)$$

or $\dfrac{x_c}{2R} = \sin\phi\cos^2\phi$, from which ϕ can readily be found by trial or by solving the cubic

$$s^3 - s + \frac{x_c}{2R} = 0 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (5a$$

where $s = \sin\phi$

In practice, as far as Tramway curves are concerned, the usua procedure is to select values for R and $\dfrac{x_c}{R}$ (which will determine ϕ) and refer to the tables for the other quantities required; it will therefore be sufficient here to show how these other qnantities can be derived from ϕ and $\dfrac{x_c}{R}$.

Taking the columns of the table (Appendix C) in order :—

From (4) $\tan\phi = 3mx_c^2$

$$m = \frac{\tan\phi}{3x_c^2}\,; \quad m\,R^2 = \frac{\tan\phi}{3\left(\frac{x_c}{R}\right)^2} \quad \dots \quad \dots \qquad (6)$$

This quantity is given in log. form for greater convenience.

$x^1 = CG = R\sin\phi$ (see Fig. 1).

$$\therefore \frac{x^1}{R} = \sin\phi \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (7)$$

$$y_c = mx_c^3 = 3\,mx_c^2 \times \frac{x_c}{3} = \frac{x_c}{3}\tan\phi$$

$$\therefore \frac{y_c}{R} = \frac{x_c}{3R} \tan \phi \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (8)$$

$$h = y_c - y^1 = y_c - R \text{ versin } \phi$$

$$\therefore \frac{h}{R} = \frac{y_c}{R} - \text{versin } \phi \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (9)$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 9m^2x^4}$$

$$s = \int \sqrt{1 + 9\, m^2x^4}\, dx$$

expanding and integrating individual terms

$$s = x_c + \frac{9}{10} m^2x^5 - \frac{9}{8} m^4x^9 + \frac{729}{208} m^6x^{13} -$$

putting $\tan \phi$ for $3mx^2$

$$s = x_c \left(1 + \frac{1}{10} \tan^2 \phi - \frac{1}{72} \tan^4 \phi + \frac{1}{208} \tan^6 \phi\right)$$

$$\frac{s}{R} = \frac{x_c}{R}\left(1 + \frac{1}{10} \tan^2 \phi - \frac{1}{72} \tan^4 \phi + \frac{1}{208} \tan^6 \phi\right) \ldots \qquad (10)$$

The last column gives the circular measure of ϕ

To find the radius of curvature at any point of the transition.

$$\rho = \frac{1}{6mx} + \frac{9}{4.} mx^3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (11)$$

will give a sufficiently accurate result.

For a fuller mathematical treatment of this curve the reader is referred to Mr. Merfield's 1895 paper.

Limits of Application.

ϕ must not be greater than one half the angle between the straights, or the circular curve will disappear and the transitions overlap. Further, as there should be no point on the transition at which the radius of curvature is less than R, the point of minimum curvature of the parabola must not be passed. This point will occur where the value of $\frac{h}{R}$ is a maximum, and may be found as follows :—

From (8) and (9)

$$\frac{h}{R} = \frac{x_c}{3R} \tan \phi - \text{versin } \phi$$

and from (4)

$$x_c = 2R \sin \phi \cos^2 \phi$$

$$\frac{h}{R} = \frac{2}{3} \sin^2 \phi \cos \phi - \text{versin } \phi$$

differentiating and equating to 0 we get

$$\sin^2 \phi = \frac{1}{6} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (12)$$

giving $\phi = 24° 5' 41{\cdot}45''$ and a corresponding value $\frac{x_c}{R} = 0{\cdot}680414$

For the method of using the tables the reader is referred to Mr. Try's paper (Appendix A).

POSITION OF SWITCHES.

At a junction the position of the switch is determined by the necessary offset at the heel, which must be equal to rail width plus groove width. This is $3\frac{1}{2}$ inches for the 80 lb. rail and $3\frac{1}{4}$ inches for the 60 lb. rail.

On a transition

$$x = \sqrt[3]{\frac{y}{m}}$$ where y is the offset at the heel of the switch.

The standard switch is 7·25 feet in length and it will be found in most cases that the tangent point is several feet in advance of the toe of the switch. As the transition curve is in abeyance until the heel of the switch is reached, it would appear that a rather longer transition is advisable where a switch occurs, but since a longer transition necessitates a flatter crossing, with more liability of breakage at the point of the frog, it is as well to keep the transition the normal length. Another objection to long transitions at junctions is that the trams cannot stop on the crossing, and consequently often have to stop a considerable distance down the street instead of at the corner.

On a circular curve

$$x = \sqrt{2Ry - y^2}$$
$$= \sqrt{0{\cdot}5833R - 0{\cdot}0851} \text{ for 80 lb. rails.}$$

The radius of the circular curve in this case must not be greater than 330 feet, since a flatter curve involves cutting away a considerable part of the flange of the rail and also leaves less room behind the rail for spiking. These switches usually occur on the inner curve in cases similar to Example (3) in Appendix A.

The Railway Commissioners avoid switches on circular curves in such cases by compounding the circular curve on the inner track with a curve of larger radius to which long transition is applied; this brings the switch on to the transition. One of the objections to this method is mentioned below.

SELECTION OF RADIUS AND LENGTH OF TRANSITION.

Owing to the varying circumstances in different cases, no rule can be laid down. In order to avoid resumption, the centre line of the inner curve must clear the kerb line by at least 8·5 feet (*i.e.*, 6 feet clearance to rail); on roads one chain or less in width this will necessitate sharp curves unless the deflection angle is small. Where the line is off the roadway flatter curves may be used.

A good length of transition to adopt is about 40 feet; this gives a rise of about one inch in 9 feet for the maximum elevation of $4\frac{1}{2}$ inches and at the same time does not unduly lengthen the curve. When pinched for room a shorter transition must be used, and in some special cases different transitions will be necessary at the two ends of the curve (*e.g.* on the inner curve at the corner of Day and Erskine Streets). In New Zealand a 30 feet transition is favoured.

Compound Circular Curves.

The method most favoured in America for gradually increasing the curvature is the "compound spiral"; that is to say, a series of short circular curves in which the radius is gradually diminished until it becomes equal to the radius of the main curve. A length of chord is chosen, aud the degrees of curvature of the successive curves must then form an arithmetical series.

For example, if we wish to connect a 100 feet curve to the straight by means of a 40 feet transition, we assume a chord length—say 8 feet—and divide the angle of curvature, in this case 60 degrees into six ($40 \div 8 + 1$) parts. The successive degrees of curvature will be 10, 20, 30, 40 and 50, coming at the end of 40 feet to the 60 degrees curve. This of course is not a true transition curve nor is it so elastic as the cubic parabola for use with sharp curves. For instance, with a 66 feet curve ($98\frac{1}{2}°$) for a 40 feet transition we must either use very short chords, which would unnecessarily complicate both the calculation and setting out of the curves, or we must have sharp changes of radius; while with a cubic parabola we can get as smooth an entrance to a sharp curve as to a flat one.

Another advantage of the cubic parobola over the so-called spiral is that, unless the chords are very short, the secant to the main curve will be longer in the case of the spiral, involving more resumption of property.

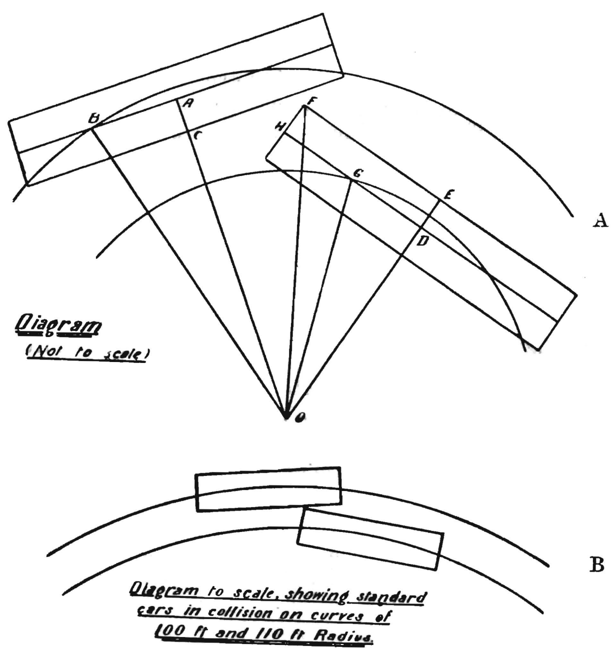

Fig. 2.

Note on the Distance Between Tracks on a Double Track Curve.

On double track lines, owing to the overhang of the cars, unless the distance between tracks be widened, the footboards of two trams passing on a curve approach closer than is considered safe; and if the curve be a sharp one the trams actually collide. This is evident from the diagram (Fig. 2b).

For the N.S.W. Tramways the increase necessary was fixed by the Railway Commissioners some years ago, the distances being tabulated for the various radii. These distances give a clearance between cars varying somewhat irregularly from $10\frac{3}{4}$ inches on a curve of 8 chains radius to $16\frac{3}{4}$ inches on a curve of 55 feet radius.

It would seem more rational to decide the clearance necessary for safety (which need not be the same for all, but might be rather greater on sharp curves), and then calculate the distance between tracks which would give this clearance. It will be sufficiently accurate if for any given radius of the inner track we find the outer radius which will just allow the cars to touch, and then add the necessary clearance. The method of calculating the outer radius is given below.

We must first know the dimensions of the car or combination of cars which will produce the worst case; for the N.S.W. Tramways this will be with the 70-seat steam car trailer on each track, as all new cars are designed to keep within the limits imposed by this case.

The dimensions are (see diagram Fig. 2a).

Length of footboard	= 2EF	= 34ft. 6in.
Width ,, ,, ...	= 2ED	= 9ft.
Distance between centres of Bogies ...	= 2DG	= 23ft. $6\frac{1}{2}$in. (23·54′).

On 10-foot tracks this car has a clearance of 1 ft. on the straight.

Referring to the diagram it will be seen that the point on the car which describes the largest circle on the inner curve is F, while the point which describes the smallest circle on the outer curve is C. These are therefore the critical points, and if O be the centre of the curves the cars will just touch when OC = OF.

Let $AB = DG = a \qquad OG = R \qquad OB = r$

$EF = l \qquad OF = R^1 \qquad OC = r^1$

$AC = DE = w$

$$\text{Now } R^1 = OF = \sqrt{OE^2 + EF^2}$$
$$= \sqrt{(OD^2 + DE^2) + EF^2}$$
$$\text{and } OD = \sqrt{OG^2 - GD^2}$$
$$\text{therefore } R^1 = \sqrt{\left\{(R^2 - a^2)^{\frac{1}{2}} + w\right\}^2 + l^2}$$
$$\text{and } r = OB = \sqrt{OA^2 + AB^2}$$
$$= \sqrt{(OC + CA)^2 + AB^2}$$
$$= \sqrt{(r^1 + w)^2 + a^2}$$

so that if $r^1 = R^1$

$$r = \sqrt{(R^1 + w)^2 + a^2}$$

Whence, given the value of R, r may be found

Expanding the formula for R^1

$$R^1 = R + w + \frac{l^2 - a^2}{2R} - \frac{wl^2}{2R^2} - \frac{l^4 - 2(a^2 + 2w^2)l^2 + a^4}{8R^3} -$$

whence

$$r = R + 2w + \frac{l^2}{2R} - \frac{wl^2 + 2wa^2 + a^2}{2R^2} - \frac{l^2(4w^2 - 1) + 4(2w + a)^2}{8R^3} -$$

For the standard car $a = 11{\cdot}77$ $a^2 = 138{\cdot}5329$
$w = 4{\cdot}5$ $w^2 = 20{\cdot}25$
$l = 17{\cdot}25$ $l^2 = 297{\cdot}5625$

$$R^1 = R + 4{\cdot}5 + \frac{159}{2R} - \frac{1339}{2R^2} - \frac{1268}{8R^3}$$

$$R^1 = R + 4{\cdot}5 + \frac{80}{R} - \frac{670}{R^2} - \frac{160}{R^3} \quad \text{approx.}$$

of which the fifth term may be omitted for all values of R over 32 feet and the fourth term for all values over 52 feet.

$$r = R + 9 + c + \frac{297{\cdot}56}{2R} - \frac{2724{\cdot}36}{2R^2} + \frac{25530{\cdot}57}{8R^3}$$

$$= R + 9 + c + \frac{149}{R} - \frac{1362}{R^2} + \frac{3191}{R^3} \quad \text{approx.}$$

Where c is the clearance required between cars.

In this formula the last term = ·02 for radii between 50 and 60 feet, ·01 from 60 to 85 feet, and may be omitted for radii over 85 feet.

For radii over 100 feet the formula

$$r = R + 9 + c + \frac{150}{R} - \frac{1400}{R^2}$$

will give a result correct to two decimal places ($\frac{1}{8}$ in.)

The following Tables show for any Radius :—the distance between tracks for which the cars would just touch; the distance adopted by the Railway Commissioners; and the difference between these to the nearest $\frac{1}{4}$ inch, which will of course be the clearance.

TABLE I.—CLEARANCES BETWEEN CARS.

Radius of Inner Curve.	Distance between Curves. Clearance Nil.	Distance adopted N S.W. Trams.		Clearance.	
feet.	feet.	ft.	in.	ft.	in.
50	11·46	12	10	1	4½
55	11·27	12	8	1	4¾
60	11·12	12	4	1	2½
66	10·95	12	2	1	2½
70	10·86	12	0	1	1½
75	10·75	11	11	1	2

Radius of Inner Curve.		Distance between Curves. Clearance Nil.		Distance adopted N.S.W. Trams.			Clearance.	
feet.		feet.		ft.	in.		ft.	in.
80	...	10·65	...	11	10	...	1	2¼
85	...	10·57	...	11	9	...	1	2
90	...	10·49	...	11	8	...	1	2
100	...	10·35	...	11	6	...	1	1¾
110	...	10·24	...	11	6	...	1	3
120	...	10·15	...	11	2	...	1	0¼
132	...	10·05	...	11	1	...	1	0¼
150	...	9·93	...	11	0	...	1	0¾
165	...	9·85	...	11	0	...	1	1¾
180	...	9·79	...	11	0	...	1	2½
198	...	9·72	...	10	10	...	1	1¼
231	...	9·62	...	10	9	...	1	1½
264	...	9·54	...	10	8	...	1	1½
297	...	9·49	...	10	7	...	1	1
330	...	9·44	...	10	6	...	1	0¾
363	...	9·40	...	10	5	...	1	0¼
396	...	9·37	...	10	4	...	0	11½
429	...	9·34	...	10	4	...	0	11¾
462	...	9·32	...	10	4	...	1	0
528	...	9·28	...	10	2	...	0	10½
875	...	9·17	...	10	0	...	0	10
1000	...	9·15	...	10	0	...	0	10¼

This Table refers only to the circular part of the curves, but, since the clearance on the straight is only 12 inches, as soon as we pass the inner tangent point we must get a clearance less than this, so that the pinch will as a rule come on the transition. However, if we so choose our transitions as to give the inner curve a lead of 20 to 30 feet from the outer (the sharper curve of course requiring the longer lead) we shall keep the clearance on the transition above 11 inches.

It may be thought that the difficulty could be avoided by increasing the distance between tracks to, say, 13 feet and keeping the lines parallel throughout, but to this there are several objections.

In the first place the roads in Sydney are mostly only 66 feet wide, of which 24 feet is footway, leaving only 42 feet of roadway. As there is no room for traffic between the lines a tramway with 10 feet centres and cars 9 feet wide leave only 11 ft. 6 in. between footboard and kirb, and this would be further diminished by any increase in the distance between tracks. In wide streets with a plantation in the middle, the trams may be run on either side of the plantation, in which case the double track becomes practically equivalent to two single tracks.

Another difficulty is of a financial nature, inasmuch as in N.S.W. the roadway between tracks is maintained by the Railway Commissioners, the remainder being maintained by the various Councils. So that every foot increase in the distance between tracks means the cost of maintenance of an extra 600 sq. yds. per mile of roadway being thrown on to the Railway Commissioners.

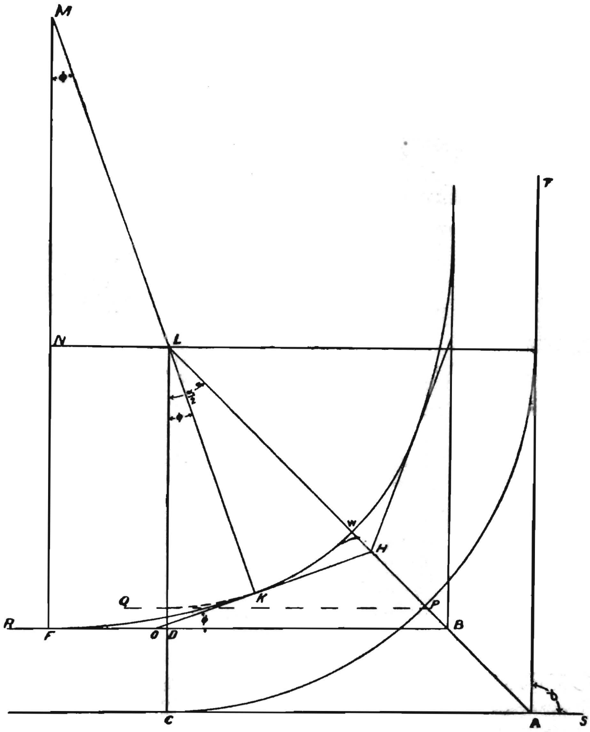

FIG. 3.

TRAMWAY CURVES (DOUBLE TRACK) WITH RADII FROM 250 FT. TO 900 FT.

As pointed out by Mr. J. C. Try, B.E., double track curves from 5 chains upward are usually set out, in the Government Tramways of N.S.W., by making the outer a simple circular curve without transition, and obtaining the necessary extra clearance on the curve by giving a transition to the inner such that h = d where d is the necessary increase in distance between tracks, the difference between the Radii of the circular curves being, of course, 10 + d. This principle may be extended to curves of about 250 ft. radius, but if the radius be less than 5 chains there must be a transition on the outer and the transition on the inner must be so chosen that $h = d + h^1$, where h^1 is the value of h for the outer curve.

For a radius of 1,000 feet or more there is sufficient clearance between cars without increasing the distance between tracks, so that two concentric curves without transition may be used.

In most cases the method mentioned above gives the inner tangent a lead of about 30 feet on the outer and involves a transition of 60 feet or more. These curves occur frequently on winding roads where the angles are small and the curve becomes almost entirely transition, while if the bends are close together the long lead will often cause an overlap, necessitating a local widening of the tracks and giving the curves an ugly appearance on the ground. There are many examples of this on the Bellevue Hill Tramway, and one very marked case on the Waverley-Bronte.

Taking Mr. Try's Example (4).

Suppose the deflection angle to be 10°. A suitable radius to adopt is 500 ft., which may be taken as the radius of the inner curve. For 500 ft. radius d = 0·3 feet, making the outer radius 510·3 ft. to give concentric curves. The outer curve need not be transitioned, but a transitioned curve having a value of h = 0·3 ft. will give the necessary clearance.

Here $h = 0{\cdot}3$

$$\frac{h}{R} = \frac{0{\cdot}3}{500} = 0{\cdot}0006$$

The nearest value of h in the Tables is ·000599 and the corresponding value of $\frac{x_c}{R}$ is 0·12 giving a transition of 60 ft.

Referring to the tables we have :—

$x_c = 60$		Offsets :—	10′	0·01
$x^1 = 30{\cdot}11$			20′	0·04
$x_c - x^1 = 29{\cdot}89$			30′	0·15
$y_c = 1{\cdot}21$			40′	0·36
$h = 0{\cdot}2995$ (say 0·3)			50′	0·70
$S = 60{\cdot}02$			60′	1·21
$\log m = 4{\cdot}747094$				

For this curve deflection angle $= 10° = \propto$

$$\phi = 3° \ 27' \ 8 \ 5''$$

$$\text{Total Tangent} = (R + h) \tan \propto/2 + x_c - x^1$$
$$= {\cdot}0874887 \times 500{\cdot}3 + 29{\cdot}89$$
$$= 73{\cdot}66$$

$$\text{Secant} = (R + h) \sec \propto/2 - R$$
$$= 1{\cdot}0038198 \times 500{\cdot}3 - 500$$
$$= 2{\cdot}21$$

$$\tfrac{1}{2} \text{ Circular arc} = R \times \text{arc} \left(\frac{\propto}{2} - \phi\right)$$
$$= 500 \times {\cdot}0270114$$
$$= 13{\cdot}50$$

$$\text{Chord of } \tfrac{1}{2} \text{ arc} = R \times \text{chord} \left(\frac{\propto}{2} - \phi\right)$$
$$= 500 \times {\cdot}0270106$$
$$= 13{\cdot}50$$

The lead of the inner tangent is $x_c - x^1$, in this case 29·89. The calculation of the outer curve need not be given as it is a simple circular curve.